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GLOBAL RIGIDITY RESULTS FOR LATTICE ACTIONS ON TORI AND NEW EXAMPLES OF VOLUME-PRESERVING ACTIONS

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ABSTRACT

Any action of a finite index subgroup in $\mathrm{SL}(n,\mathbb{Z})$, $n \geq 4$ on the n-dimensional torus which has a finite orbit and contains an Anosov element which splits as a direct product is smoothly conjugate to an affine action. We also construct first examples of real-analytic volume-preserving actions of $\mathrm{SL}(n,\mathbb{Z})$ and other higher-rank lattices on compact manifolds which are not conjugate (even topologically) to algebraic models.

1. Introduction

This paper can considered as a sequel to [K-L]. It is a part of a broad program directed toward understanding the dynamics of "sufficiently large" smooth group actions on compact manifolds. These actions display remarkable rigidity phenomena. For a sample of results related to different aspects of the program, we refer the reader to [Z1], [K], [K-S1], [K-S2] [P-Y], [Gr], [Gh], [H2], and [K-L-Z].

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Here we will consider only one aspect of this program. Namely, let Γ be an irreducible lattice in a semisimple Lie group of rank ≥ 2. At the 1984 MSRI conference [H1], and again in his I.C.M. address [Z1], Zimmer posed the following general question: What are the smooth ergodic actions of Γ on compact manifolds which preserve a smooth volume form? In particular, he lists three basic classes of such actions:

(i) Isometric actions.

(ii) Γ acts on $M = H/\Lambda$ via ρ, where $\Gamma \subset G$ and $\Lambda \subset H$ are lattices, with Λ co-compact, and $\rho: G \to H$ is a homomorphism.

(iii) Γ acts on $M = N/\Lambda$, where Λ is a (necessarily co-compact) lattice in a nilpotent Lie group N, and Γ is a lattice in G, where G is a semisimple group of automorphisms of N, such that Γ preserves Λ.

All previously known examples of smooth, volume-preserving, ergodic Γ-actions were obtained from this list by simple algebraic constructions such as products, suspensions, and finite extensions.

From the point of view of dynamics, isometric actions are very different from the other two types; the latter display both certain similarities (such as the presence of hyperbolic or partially-hyperbolic elements) and essential differences (e.g., examples of type (iii) always preserve a rational structure and in particular have a dense set of finite orbits, while for examples of type (ii) this is generally not the case). However, all three types are algebraic in that they preserve a rigid geometric structure in the sense of Gromov [Gr]: i.e., a Riemannian metric in case (i), a homogeneous H-structure in case (ii), and an affine connection in case (iii), and the same is true for the algebraic modifications mentioned above.

In Section 4, we construct some new examples of real-analytic, volume-preserving, ergodic lattice actions (with a dense set of finite orbits) which do not preserve a rigid geometric structure. However, such a structure does exist on an invariant, open, dense set. This suggests the following general conjecture:

CONJECTURE 1.1: *Every smooth, volume-preserving, ergodic action of an irreducible lattice Γ in a semisimple Lie group of rank ≥ 2 preserves a rigid geometric structure on a Γ-invariant, open, dense set.*

One important implication of our examples is that some additional hypothesis (for example, hyperbolicity or partial hyperbolicity for some element(s)) is needed to establish conjugacy with an algebraic model.

In Sections 2 and 3, we establish a fairly definitive global rigidity result for a basic example of type (iii), namely, the action of a finite-index subgroup in $\mathbf{SL}(n,\mathbb{Z})$ on $\mathbb{T}^n$, $n \geq 4$ (Theorem 2.1). The two assumptions we need are the presence of a finite orbit and the existence of a "split" Anosov element (cf. 2.1, (ii)). (Both of these conditions are open in the C^1-topology; for the first condition, this follows by a theorem of Stowe [St], together with H^1-vanishing, cf. [K-L].) It is probably worth remarking that we make no *a priori* assumption about the existence of an invariant volume form, or even an absolutely continuous measure.

The proof of (2.1) is divided into two components. In Section 2, we establish topological conjugacy under the above hypotheses, by a refinement of the method developed in our previous paper [K-L]. This method makes extensive use of the rational (periodic) structure for the linear action, together with purely group-theoretic information about the structure of Γ, such as the congruence subgroup property and the Margulis finiteness theorem. The central idea is to tie together this algebraic information with the orbit structure for the action via hyperbolic dynamics (the Franks–Manning classification of Anosov maps on tori and the description of their centralizers).

In Section 3, we show that any Γ-action which is topologically conjugate to a linear action and contains an Anosov element is in fact smoothly conjugate. (This generalizes an earlier result established by Hurder [H2], who required a collection of n commuting Anosov elements with associated 1-dimensional strong stable foliations such that the 1-dimensional subspaces span the tangent space at every point.) The two main ingredients in the argument are first, the application of Margulis's superrigidity theorem to the isotropy representation of the stabilizer at each periodic point, and second, the Livshitz–Sinai criterion for the existence of smooth invariant measures for Anosov diffeomorphisms. The main point is that in the presence of an Anosov element, the finite-dimensional periodic data must "glue together" to yield a continuous reduction of the derivative cocycle to the cocycle defined by the homomorphism $\Gamma \to \mathbf{GL}(n,\mathbb{Z})$ corresponding to the action on homology.

Our methods are applicable to some other lattice actions on tori (e.g., a subgroup of finite index in $\mathbf{Sp}(n,\mathbb{Z})$, $n \geq 3$, acting on $\mathbb{T}^{2n}$), as well as some special cases of type (ii).

An alternative approach to both local and global rigidity for volume-preserving actions is based on Zimmer's superrigidity theorem for cocycles together with

Kazhdan's property T [K-L-Z]. Two advantages of this method are that it is applicable to actions of lattices in groups of rank 2, in particular to actions of subgroups of finite index in $\mathbf{SL}(3,\mathbb{Z})$ on $\mathbb{T}^3$, and that it does not require the existence of an invariant rational structure. The two techniques are complementary and parallel, in that the first requires the existence of a finite orbit, while the second requires an absolutely continuous invariant measure. Since the existence of a periodic orbit is equivalent to the existence of an invariant atomic measure, the hypotheses may be viewed as extreme cases of the more general hypothesis that the action preserves a non-trivial finite measure. This observation very naturally leads to a question whether the latter hypothesis is sufficient for rigidity under proper hyperbolicity assumptions. The remaining case is that when the measure in question is singular non-atomic. A posteriori, such measures do not exist since for algebraic actions the only invariant measures are combinations of atomic and Lebesgue. As shown in [K-L-Z] rigidity for actions pereserving non-atomic measures holds under a somewhat stronger hyperbolicity assumption.

The first author discovered the basic construction described in Section 4 while visiting I.H.E.S., and he would like to thank the faculty there for providing an ideal working environment. The second author prepared portions of this paper while visiting I.M.P.A. and the mathematics department at the Pennsylvania State University, and he would like to thank the faculties at both institutions for their hospitality; he would also like to acknowledge especially helpful conversations with D. Ramakrishnan and J. C. Yoccoz.

2. Topological conjugacy

The main rigidity result in this paper is the following˙

THEOREM 2.1: *Suppose Γ is a subgroup of finite index in $\mathbf{SL}(n,\mathbb{Z})$, $n \geq 4$, $M = \mathbb{T}^n$, and $\rho \in R(\Gamma, \mathrm{Diff}(M))$ such that*

(i) *there exists a fixed point, i.e., there exists $x_0 \in M$ such that $\rho(\gamma)x_0 = x_0$ for every $\gamma \in \Gamma$,*

(ii) *there exists a direct-sum decomposition of $\mathbb{Q}^n$ as a vector space over $\mathbb{Q}$,*

$$\mathbb{Q}^n = V_1 \oplus V_2,\ V_1 \simeq \mathbb{Q}^k,\ V_2 \simeq \mathbb{Q}^\ell,\ k+\ell = n,\ k,\ell \geq 2,$$

and an element $\lambda_0 \in \Lambda = \{\gamma \in \Gamma \mid \gamma V_i = V_i,\ i = 1,2\}$ such that the diffeomorphism $\rho(\lambda_0)$ is Anosov.

Let $\rho_*: \Gamma \to \mathbf{GL}(n, \mathbb{Z})$ denote the homomorphism corresponding to the action on $H_1(M) \simeq \mathbb{Z}^n$. Then ρ is smoothly equivalent to the linear action corresponding to ρ_*; i.e., there exists a diffeomorphism h of M, homotopic to the identity, such that $\rho(\gamma) = h\rho_*(\gamma)h^{-1}$ for every $\gamma \in \Gamma$.

Remarks: 1. To make our result completely parallel to that of [K-L-Z] for actions perserving an absolutely continuous measure we would have to replace (ii) by a weaker assumption of existence of *some* (not necesserily split) Anosov element in the action.

2. Under conditions of the theorem the action of the whole group Γ is diffeomorphically equivalent to an *affine* action which may not have a fixed point. See the comment before Lemma 2.15 and [H3].

In this section we prove the topological version of (2.1); namely, the existence of a *homeomorphism* h, homotopic to the identity, conjugating ρ and ρ_*. Before beginning the proof itself, it will be convenient to establish some purely algebraic results and notation. For each $m \in \mathbb{N}^+$, let $\Gamma(m) = \mathbf{SL}(n, \mathbb{Z})_m = \{\gamma \in \mathbf{SL}(n, \mathbb{Z}) | \ \gamma \equiv I \bmod m\}$ denote the principal congruence subgroup mod m in $\mathbf{SL}(n, \mathbb{Z})$. A subgroup in $\mathbf{SL}(n, \mathbb{Z})$ is called a **congruence subgroup** if it contains $\Gamma(m)$ for some m. The following celebrated theorem of Mennicke and Bass–Lazard–Serre [B-M-S] appeared as (5.3) in [K-L].

PROPOSITION 2.2: *For $n \geq 3$, every subgroup of finite index in $\mathbf{SL}(n, \mathbb{Z})$ is a congruence subgroup.*

LEMMA 2.3: *Suppose $A_0 \in \mathbf{SL}(n, \mathbb{C})$ such that $A_0\Gamma(m)A_0^{-1} \subset \mathbf{SL}(n, \mathbb{Z})$. Then there exists a scalar matrix $\lambda \in \mathbf{GL}(n, \mathbb{C})$ and $A \in \mathbf{GL}(n, \mathbb{Q})$ such that $A_0 = \lambda A$. Conversely, if $A \in \mathbf{GL}(n, \mathbb{Q})$ and $m \in \mathbb{N}^+$, then there exists $m' \in \mathbb{N}^+$ such that $A\Gamma(m')A^{-1} \subset \Gamma(m)$.*

Proof: The second assertion is obvious; simply take $m' = md^2$, where d is a common denominator for the entries in A and A^{-1}.

To establish the converse, set $\mathbf{G} = \mathbf{SL}(n, \mathbb{C})/Z(\mathbf{SL}(n, \mathbb{C}))$, and let $\pi: \mathbf{SL}(n, \mathbb{C}) \to \mathbf{G}$ denote the projection. Recall that $\mathbf{G}$ has a natural $\mathbb{Q}$-structure, which is most easily realized via the adjoint representation on $\mathfrak{sl}(n, \mathbb{C})$.

Our hypothesis on A_0 implies that $\pi(A_0) \in \mathrm{Comm}_{\mathbf{G}}(\mathbf{G}_{\mathbb{Z}})$, where $\mathrm{Comm}_G(\Gamma) = \{g \in G | \ g\Gamma g^{-1}$ and Γ are commensurable$\}$. It follows from a general theorem due to Borel (Proposition 6.2.2 in [Z2]) that $\mathrm{Comm}_{\mathbf{G}}(\mathbf{G}_{\mathbb{Z}}) = \mathbf{G}_{\mathbb{Q}}$, hence $\pi(A_0) \in \mathbf{G}_{\mathbb{Q}}$. Equivalently, conjugation by A_0 defines a $\mathbb{Q}$-rational automorphism of $\mathbf{SL}(n, \mathbb{Q})$.

Let $\mathbf{T}$ denote the standard maximal $\mathbb{Q}$-split torus in $\mathbf{SL}(n, \mathbb{C})$, i.e., the diagonal subgroup. Then $A_0\mathbf{T}A_0^{-1}$ is again a maximal $\mathbb{Q}$-split torus in $\mathbf{SL}(n, \mathbb{C})$, and there exists $B \in \mathbf{SL}(n, \mathbb{Q})$ such that $B\mathbf{T}B^{-1} = A_0\mathbf{T}A_0^{-1}$, i.e., so that $B^{-1}A_0 \in N(\mathbf{T})$, the normalizer of $\mathbf{T}$ in $\mathbf{SL}(n, \mathbb{C})$. Then there exists a permutation matrix $S \in \mathbf{GL}(n, \mathbb{Q})$ so that $SB^{-1}A_0 \in Z(\mathbf{T}) = \mathbf{T}$. Finally, it is obvious that any diagonal matrix in $\pi^{-1}(\mathbf{G}_{\mathbb{Q}})$ must be a scalar multiple of a matrix in $\mathbf{GL}(n, \mathbb{Q})$. ∎

The following well-known observation will play an essential role below, as it did in [K-L]; a proof appears under Proposition 0 of [P-Y] (cf. also the proof of (2.7) below).

Lemma 2.4: *Suppose that $A \in \mathbf{SL}(n, \mathbb{Z})$ is a hyperbolic linear automorphism of $\mathbb{T}^n$, and $\varphi: \mathbb{T}^n \to \mathbb{T}^n$ is a homeomorphism which fixes the origin and commutes with A. Then $\varphi \in \mathbf{GL}(n, \mathbb{Z})$ is linear.*

The first step in the proof of (2.1) is to reduce to the case in which $\rho_*: \Gamma \to \mathbf{GL}(n, \mathbb{Z})$ is simply the inclusion $\rho_*(\gamma) = \gamma$.

Proposition 2.5: *There exists a subgroup $\Gamma' \subset \Gamma$ of finite index (at most two) and a matrix $A \in \mathbf{GL}(n, \mathbb{Q})$ such that A conjugates the restriction $\rho_* \,|\, \Gamma': \Gamma' \to \mathbf{GL}(n, \mathbb{Z})$ to either the identity or the Cartan involution (inverse transpose). That is, either $\rho_*(\gamma) = \iota(\gamma)A\gamma A^{-1}$ for every $\gamma \in \Gamma$ or $\rho_*(\gamma) = \iota(\gamma)A(\gamma^{-1})^t A^{-1}$ for every $\gamma \in \Gamma$, where $\iota: \Gamma \to \{\pm I\}$ is a homomorphism with kernel Γ'.*

Proof: Passing to a subgroup of finite index (at most two), we may assume that $\rho_*(\Gamma) \subset \mathbf{SL}(n, \mathbb{Z})$. Consider the induced homomorphism $\varphi: \Gamma \to \mathbf{G}_{\mathbb{R}}$ (where $\mathbf{G}$ denotes the $\mathbb{Q}$-group $\mathbf{SL}(n, \mathbb{C})/Z(\mathbf{SL}(n, \mathbb{C}))$, as above). Set $\mathbf{H} \subset \mathbf{G}$ equal to the Zariski closure of $\varphi(\Gamma)$. By (3.1.8) in [Z2], $\mathbf{H}$ is defined over $\mathbb{R}$ (in fact over $\mathbb{Q}$). Passing to the finite-index subgroup $\varphi^{-1}(\mathbf{H}^\circ)$, we may assume that $\mathbf{H}$ is connected. Now set $\bar{\mathbf{H}}$ equal to the quotient obtained by first dividing $\mathbf{H}$ by its radical, then dividing the resulting group by its center. $\bar{\mathbf{H}}$ is a connected semisimple $\mathbb{Q}$-group with trivial center. Write $\bar{\mathbf{H}} = \mathbf{H}' \times \mathbf{H}''$, where $\mathbf{H}''_{\mathbb{R}}$ is compact and $(\mathbf{H}_i)_{\mathbb{R}}$ is non-compact for every simple factor $\mathbf{H}_i$ of $\mathbf{H}'$. Then by Margulis's superrigidity theorem (Theorem 5.1.2 of [Z2]), the induced homomorphism $\varphi': \Gamma \to \mathbf{H}'_{\mathbb{R}}$ extends to an $\mathbb{R}$-rational homomorphism $\mathbf{SL}(n, \mathbb{C}) \to \mathbf{H}'$. Examining the list of finite-dimensional $\mathbb{C}$-linear representations of $\mathfrak{sl}(n, \mathbb{C})$, we see that there are only two possibilities: either $\mathbf{H}'$ is trivial or $\mathbf{H}' = \bar{\mathbf{H}} = \mathbf{H} = \mathbf{G}$.

Now if $\mathbf{H}'$ were trivial, $\rho_*(\Gamma)$ would be contained in a compact extension of a solvable group, hence $\rho_*(\Gamma)$ would be amenable. But Γ, and therefore $\rho_*(\Gamma)$, are Kazhdan, and any amenable Kazhdan group is compact. Thus $\rho_*(\Gamma)$ would have to be finite. But by a theorem of Manning [M], any Anosov diffeomorphism of M acts by a hyperbolic automorphism on H_1. Thus the first possibility is excluded since $\rho(\lambda_0)$ is Anosov, and φ extends to an $\mathbb{R}$-rational homomorphism $\mathbf{SL}(n,\mathbb{C}) \to \mathbf{G}$. Returning to the list of representations of $\mathfrak{sl}(n,\mathbb{C})$, we see that after passing once again to a subgroup of index at most two and conjugating by some matrix $A_0 \in \mathbf{SL}(n,\mathbb{C})$, the only possibilities for ρ_* are the identity map and the Cartan involution $\gamma \mapsto (\gamma^{-1})^t$.

We have shown that there exists a subgroup $\Gamma' \subset \Gamma$ of index at most four such that either $\rho_*(\gamma) = A_0\gamma A_0^{-1}$ or $\rho_*(\gamma) = A_0(\gamma^{-1})^t A_0^{-1}$ for every $\gamma \in \Gamma'$, where $A_0 \in \mathbf{SL}(n,\mathbb{C})$ such that $A_0\Gamma' A_0^{-1} \subset \mathbf{SL}(n,\mathbb{Z})$. By (2.2), $\Gamma' \supset \Gamma(m)$ for some $m \in \mathbb{N}^+$, so by (2.3), we can write $A_0 = \lambda A$, with $\lambda \in \mathbf{GL}(n,\mathbb{C})$ scalar and $A \in \mathbf{GL}(n,\mathbb{Q})$.

Finally, it is well-known (and easy to see) that the centralizer of any subgroup of finite index in $\mathbf{SL}(n,\mathbb{Z})$ in $\mathbf{GL}(n,\mathbb{R})$ is equal to the subgroup of scalar matrices, and the only scalars in $\mathbf{GL}(n,\mathbb{Z})$ are $\pm I$. Γ' has a subgroup Γ_0 of finite index which is normal in Γ. Then it is easy to see that any homomorphism $\Gamma \to \mathbf{GL}(n,\mathbb{Z})$ which restricts to the identity (or the Cartan involution) on Γ_0 can differ from the corresponding homomorphism on Γ only by a homomorphism $\iota\colon \Gamma \to \{\pm I\}$. ∎

One immediate consequence of (2.5) is that $\rho_*(\gamma)$ is hyperbolic for every hyperbolic matrix $\gamma \in \Gamma$. This makes it possible to establish the following

LEMMA 2.6: *Suppose that there exists a subgroup $\Gamma' \subset \Gamma$ of finite index such that the restriction $\rho \mid \Gamma'$ is topologically equivalent to the linear action corresponding to the restriction $(\rho_* \mid \Gamma')\colon \Gamma' \to \mathbf{GL}(n,\mathbb{Z})$ by a homeomorphism h of M such that $h(0) = x_0$. Then ρ itself is topologically equivalent to the linear action corresponding to ρ_*.*

Proof: By (5.2) of [K-L], Γ is generated by a finite collection of hyperbolic matrices, say $\{\gamma_1,\dots,\gamma_r\}$. By hypothesis, there is a homeomorphism h of M, homotopic to the identity, such that $h(0) = x_0$ and $\rho(\gamma) = h\rho_*(\gamma)h^{-1}$ for every $\gamma \in \Gamma'$. Since each γ_i has infinite order and Γ' is of finite index, there exist $n_i \in \mathbb{N}^+$, $1 \le i \le r$, such that $\gamma_i^{n_i} \in \Gamma'$.

Obviously the diffeomorphism $h^{-1}\rho(\gamma_i)h$ commutes with the hyperbolic linear automorphism $h^{-1}\rho(\gamma_i^{n_i})h = \rho_*(\gamma_i)^{n_i}$ and $h^{-1}\rho(\gamma_i)h(0) = 0$, so by (2.4),

$h^{-1}\rho(\gamma_i)h$ is a linear map. But since h is homotopic to the identity, it follows by comparing the actions on H_1 that $h^{-1}\rho(\gamma_i)h = \rho_*(\gamma_i)$. ∎

Note that the hyperbolic matrix λ_0 appearing in condition (ii) of (2.1) has infinite order, so that given any subgroup Γ' of finite index in Γ, there exists $N \in \mathbb{N}^+$ such that $\lambda_0^N \in \Gamma'$. Obviously $\rho(\lambda_0^N) = \rho(\lambda_0)^N$ is Anosov. Thus Γ' satisfies the same hypotheses as Γ, and we may pass freely to subgroups of finite index without loss of generality.

Now take Γ' and $A \in \mathbf{GL}(n, \mathbb{Q})$ as in (2.5). Fix $m \in \mathbb{N}^+$ so that $A^{-1}\Gamma(m)A \subset \Gamma'$ by (2.3) and define $\rho' \in R(\Gamma(m), \mathrm{Diff}(M))$ by setting $\rho'(\gamma) = \rho(A^{-1}\gamma A)$ or $\rho'(\gamma) = \rho(A^{-1}(\gamma^{-1})^t A)$, respectively. Then $\rho'_*(\gamma) = \gamma$ for every $\gamma \in \Gamma(m)$, and

$$\rho'(\gamma) = h\gamma h^{-1} \implies \rho(\gamma) = h\rho_*(\gamma)h^{-1}.$$

Thus we may assume without loss of generality that $\rho_*(\gamma) = \gamma$ for every $\gamma \in \Gamma$.

Conjugating ρ by a translation, we may assume that the fixed point x_0 equals the origin, so that ρ lifts uniquely to an action $\tilde{\rho} \in R(\Gamma, \mathrm{Diff}(\mathbb{R}^n))$ such that $\tilde{\rho}(\gamma)(0) = 0$ and $\tilde{\rho}(\gamma)(x + z) = \tilde{\rho}(\gamma)(x) + \gamma z$ for every $\gamma \in \Gamma$, $x \in \mathbb{R}^n$, and $z \in \mathbb{Z}^n$.

Lemma 2.7: *Suppose that the lift of ρ to some finite cover is topologically equivalent to the standard linear action. Then ρ is topologically equivalent to the standard linear action.*

Proof: The hypothesis is equivalent to the existence of a homeomorphism $\tilde{h}$ of $\mathbb{R}^n$ and $N \in \mathbb{N}^+$ such that $\tilde{\rho}(\gamma) = \tilde{h}\gamma\tilde{h}^{-1}$ and $\tilde{h}(x + Nz) = \tilde{h}(x) + Nz$ for every $\gamma \in \Gamma$, $x \in \mathbb{R}^n$, and $z \in \mathbb{Z}^n$. Then the claim is that in fact $\tilde{h}(x + z) = \tilde{h}(x) + z$ for every $z \in \mathbb{Z}^n$.

Fix a hyperbolic element $\gamma_0 \in \Gamma$. Since

$$\tilde{h}\gamma_0\tilde{h}^{-1}(x + z) = \tilde{h}\gamma_0\tilde{h}^{-1}(x) + \gamma_0 z \quad \forall x \in \mathbb{R}^n,\ z \in \mathbb{Z}^n,$$
$$\tilde{h}\gamma_0^i\tilde{h}^{-1}(x + z) = \tilde{h}\gamma_0^i\tilde{h}^{-1}(x) + \gamma_0^i z \quad \forall x \in \mathbb{R}^n,\ z \in \mathbb{Z}^n,\ i \in \mathbb{Z}.$$

Since

$$\tilde{h}(x + Nz) = \tilde{h}(x) + Nz \quad \forall x \in \mathbb{R}^n,\ z \in \mathbb{Z}^n,$$

there exists a finite constant $\delta > 0$ such that

$$d(\tilde{h}(x), x) < \delta \quad \forall x \in \mathbb{R}^n.$$

Thus

$$d(\gamma_0^i\tilde{h}^{-1}(x + z), \gamma_0^i(\tilde{h}^{-1}(x) + z)) < 2\delta \quad \forall i \in \mathbb{Z}.$$

But this implies that $\tilde{h}^{-1}(x+z) = \tilde{h}^{-1}(x) + z$, since γ_0 is hyperbolic. Thus $\tilde{h}(x+z) = \tilde{h}(x) + z$, as claimed. ∎

We are now in a position to standardize condition (ii) in (2.1). In particular, given $V_1 \simeq \mathbb{Q}^k$ and $V_2 \simeq \mathbb{Q}^\ell$, there exists a matrix $A \in \mathbf{SL}(n, \mathbb{Q})$ such that $AV_i = W_i$, $i = 1, 2$,

$$W_1 = \mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_k, \quad W_2 = \mathbb{Q}e_{k+1} \oplus \cdots \oplus \mathbb{Q}e_n,$$

respectively, where e_i denotes the i^{th} standard basis vector. For large enough $m \in \mathbb{N}^+, \Gamma(m)$, $A^{-1}\Gamma(m)A \subset \Gamma$. Let $\tilde{\sigma} \in R(\Gamma(m), \mathrm{Diff}(\mathbb{R}^n))$, $\tilde{\sigma}(\gamma) = A\tilde{\rho}(A^{-1}\gamma A)A^{-1}$ Fix a common denominator $N \in \mathbb{N}^+$ for the entries in A^{-1}. Then $\tilde{\sigma}$ satisfies

$$\tilde{\sigma}(\gamma)(x + Nz) = \tilde{\sigma}(\gamma)(x) + Nz \quad \forall x \in \mathbb{R}^n, \; z \in \mathbb{Z}^n.$$

In other words, $\tilde{\sigma}$ descends to an action σ on a suitable finite cover, and the corresponding action on H_1 satisfies $\sigma_*(\gamma) = \gamma$ for every $\gamma \in \Gamma(m)$. Then if h is a homeomorphism of $\mathbb{T}^n$, homotopic to the identity, such that $h(0) = 0$ and $\sigma(\gamma) = h\gamma h^{-1}$, we have $\tilde{\rho}(\gamma) = A^{-1}\tilde{h}A\gamma A^{-1}\tilde{h}^{-1}A$, where $\tilde{h}$ is the unique lift of h to a homeomorphism of $\mathbb{R}^n$ such that $\tilde{h}(0) = 0$. But $A^{-1}\tilde{h}A$ descends to a conjugating homeomorphism for ρ on a suitable finite cover, so we can apply (2.7). Also, for large enough $m', N \in \mathbb{N}^+$, $A\Gamma(m')A^{-1} \subset \Gamma(m)$, $\lambda_0^N \in \Gamma(m')$, and $\sigma(A\lambda_0^N A^{-1})$ is smoothly conjugate to a lift of $\rho(\lambda_0)^N$ and is therefore Anosov. Thus we may assume without loss of generality that the rational subspaces V_i in condition (ii) of (2.1) are in standard position, i.e., that

$$V_1 = \mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_k \text{ and } V_2 = \mathbb{Q}e_{k+1} \oplus \cdots \oplus \mathbb{Q}e_n.$$

We can summarize the reductions we have made and establish notation as follows. We suppose $\Gamma = \Gamma(m)$, the principal congruence subgroup mod m. The origin is a fixed point for the action ρ, i.e., $\rho(\gamma)(0) = 0$ for every $\gamma \in \Gamma$, the homomorphism $\rho_*\colon \Gamma \to \mathbf{SL}(n, \mathbb{Z})$ corresponding to the action of Γ on H_1 is simply the inclusion, $\gamma \mapsto \gamma$, and there exists a hyperbolic element $\lambda_0 \in \Lambda$ (where Λ is the subgroup which preserves the standard splitting $V_1 \oplus V_2$, as above) such that $\rho(\lambda_0)$ is Anosov.

Set

$$\Lambda_1 = \{\gamma \in \Gamma \mid \gamma V_i = V_i, \; i = 1, 2, \text{ and } \gamma|_{V_1} = \mathrm{Id}_{V_1}\} \simeq \mathbf{SL}(\ell, \mathbb{Z})_m,$$
$$\Lambda_2 = \{\gamma \in \Gamma \mid \gamma V_i = V_i, \; i = 1, 2, \text{ and } \gamma|_{V_2} = \mathrm{Id}_{V_2}\} \simeq \mathbf{SL}(k, \mathbb{Z})_m.$$

Note that $\Lambda = \Lambda_1 \times \Lambda_2$. Since there is little danger of confusion, we denote by V_i as well the corresponding subspaces in $\mathbb{R}^n$, so that

$$V_1 = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_k \text{ and } V_2 = \mathbb{R}e_{k+1} \oplus \cdots \oplus \mathbb{R}e_n.$$

Let $\pi\colon \mathbb{R}^n \to \mathbb{T}^n$ denote the natural projection and set $T_i = \pi(V_i)$, $i = 1, 2$, so that T_1 is the fixed-point set for the linear action of Λ_1, and T_2 for Λ_2. The key ingredient in the proof of (2.1) is the following

LEMMA 2.8 (cf. (3.4) in [K-L]): *There exists a homeomorphism h of $\mathbb{T}^n$, homotopic to the identity, such that $h(0) = 0$, and setting $\bar{T}_i = hT_i$, $\bar{T}_i \subset \{x \in \mathbb{T}^n \mid \rho(\gamma)x = x\ \forall \gamma \in \Lambda_i\}$, $i = 1, 2$.*

Proof: By a theorem of Franks [F], the Anosov diffeomorphism $\rho(\lambda_0)$ is conjugate to the linear action of $\lambda_0 = \rho_*(\lambda_0)$ by a homeomorphism h, homotopic to the identity, such that $h(0) = 0$. Set $\bar{T}_i = hT_i$, $i = 1, 2$.

Write $\lambda_0 = (\alpha_0\beta_0)$, where $\alpha_0 \in \Lambda_1$ and $\beta_0 \in \Lambda_2$ are hyperbolic. Then since $h^{-1}\rho(\alpha_0)h$ and $h^{-1}\rho(\lambda_0)h = \lambda_0$ commute and $h^{-1}\rho(\alpha_0)h(0) = 0$, it follows from (2.4) that $h^{-1}\rho(\alpha_0)h = \alpha_0$.

Then if $\beta \in \Lambda_2$ is any element whatsoever, $h^{-1}\rho(\beta)h$ and $h^{-1}\rho(\alpha_0)h = \alpha_0$ commute and $h^{-1}\rho(\beta)h(0) = 0$, so that $h^{-1}\rho(\beta)h$ must preserve T_2, which may be characterized as the closure of the stable (or of the unstable) manifold for α_0 through 0. Then since $\alpha_0 \,|\, T_2$ is hyperbolic, it follows by another application of (2.4) that $h^{-1}\rho(\beta)h \,|\, T_2 = \mathrm{Id}_{T_2}$. Hence $\rho(\beta) \,|\, \bar{T}_2 = \mathrm{Id}_{\bar{T}_2}$. An analogous argument shows that $\rho(\alpha)(x) = x$ for every $\alpha \in \Lambda_1$ and $x \in \bar{T}_1$. ∎

LEMMA 2.9: *For each $N \geq 2$ set*

$$\lambda_N = \begin{pmatrix} I_{k-1} & & & \\ & 1 & m & \\ & Nm & Nm^2+1 & \\ & & & I_{\ell-1} \end{pmatrix} \in \Gamma.$$

Then the subgroup Ξ_N generated by Λ_1 together with $\lambda_N \Lambda_2 \lambda_N^{-1}$ is a subgroup of finite index in Γ.

Proof: Establish notation as follows. For $1 \leq i, j \leq n$, let $e_{i,j}$ denote the matrix with 1 as its $(i,j)^{\text{th}}$ entry and all remaining entries 0. Then for $z \in \mathbb{Z}$ and $i \neq j$ let $E_{i,j}(z)$ denote the elementary matrix $E_{i,j}(z) = I + z\, e_{i,j} \in \mathbf{SL}(n, \mathbb{Z})$. As usual, for $A, B \in \mathbf{SL}(n, \mathbb{Z})$, we denote by $[A, B] = ABA^{-1}B^{-1}$ the commutator. For

$1 \leq i < k$ and $k+1 \leq j < n$,

$$\begin{aligned}
[\lambda_N E_{i,k}(m)\lambda_N^{-1}, E_{k+1,n}(m)] &= E_{i,n}(-m^3),\\
[E_{n,k+1}(m), \lambda_N E_{k,i}(m)\lambda_N^{-1}] &= E_{n,i}(Nm^3),\\
[E_{i,n}(m^3), E_{n,j}(m)] &= E_{i,j}(m^4), \text{ and}\\
[E_{j,n}(m), E_{n,i}(Nm^3)] &= E_{j,i}(Nm^4).
\end{aligned}$$

Then for $k+1 \leq j \leq n$,

$$\begin{aligned}
\lambda_N E_{1,k}(m^3)\lambda_N^{-1} E_{1,k+1}(m^4) &= E_{1,k}((Nm^2+1)m^3),\\
E_{k+1,1}(-Nm^4)\lambda_N E_{k,1}(m^3)\lambda_N^{-1} &= E_{k,1}(m^3),\\
[E_{k,1}(m^3), E_{1,j}(m^4)] &= E_{k,j}(m^7), \text{ and}\\
[E_{j,1}(Nm^4), E_{1,k}((Nm^2+1)m^3)] &= E_{j,k}((Nm^2+1)Nm^7).
\end{aligned}$$

In particular, $E_{i,j}((Nm^2+1)Nm^7) \in \Xi_N$ for every $1 \leq i \neq j \leq n$. Thus by an observation due to Vaserstein [V] (cf. (5.8) in [K-L]), $\Xi_N \supset \Gamma(((Nm^2+1)Nm^7)^2)$. ∎

In order to complete the argument, we need a collection of Anosov generators. (The fact that the generating set is finite is actually irrelevant.)

LEMMA 2.10: *There exists a subgroup $\Gamma' \subset \Gamma$ of finite index such that Γ' has a finite generating set $\{\gamma_1, \cdots, \gamma_r\}$ such that each $\rho(\gamma_i)$ is Anosov.*

Proof: Set Γ' equal to the normal subgroup generated by λ_0, i.e., Γ' is the smallest normal subgroup containing γ_0. By a theorem of Margulis and Kazhdan (Theorem 8.1.2 in [Z2]), any infinite normal subgroup in Γ is of finite index. Any lattice in a connected Lie group is finitely-generated; for lattices in groups of $\mathbb{R}$-rank ≥ 2 this follows from Kazhdan's property T §7.1 of [Z2]. It is easy to see that any generating set in a finitely-generated group contains a finite subset which generates. Since the collection $\{\gamma\lambda_0\gamma^{-1} \mid \gamma \in \Gamma\}$ obviously generates Γ' and each $\rho(\gamma\lambda_0\gamma^{-1}) = \rho(\gamma)\rho(\lambda_0)\rho(\gamma)^{-1}$ is Anosov, the lemma follows. ∎

Appealing once again to the theorem of Franks, we obtain homeomorphisms h_i homotopic to the identity, such that $h_i(0) = 0$ and $\rho(\gamma_i) = h_i\gamma_i h_i^{-1}$, $1 \leq i \leq r$.

For each $N \geq 2$, set $T_2^N = \lambda_N T_2 \subset \{x \in \mathbb{T}^n \mid \rho(\gamma)x = x \;\forall \gamma \in \lambda_N \Lambda_2 \lambda_N^{-1}\}$. Note that T_1 and T_2^N intersect in Nm^2+1 points, evenly distributed along the circle $C \subset \mathbb{T}^n$ which is the image under π of the k^{th} coordinate axis.

LEMMA 2.11: *Let* $\{0 = q_0, \dots, q_{Nm^2}\} = T_1 \cap T_2^N$. *Then the* h_i*'s agree on* $T_1 \cap T_2^N$, *i.e.,* $h_1(q_i) = \cdots = h_r(q_i)$, $0 \le i \le Nm^2$.

Proof: ((3.9) in [K-L]) As above, we denote by $\tilde{\rho}(\gamma)$, $\tilde{h}_i$ the unique lifts to maps $\mathbb{R}^n \to \mathbb{R}^n$ which fix the origin.

The pre-image $\pi^{-1}(T_1) \subset \mathbb{R}^n$ is a countable collection of k-planes parallel to V_1. For $z \in \mathbb{Z}^\ell$, denote by R_z the component of $\pi^{-1}(T_1)$ through the point $\binom{0}{z} \in \mathbb{Z}^n \subset \mathbb{R}^n$;

$$R_z = \{x \in \mathbb{R}^n \mid \gamma x = x + \gamma\tbinom{0}{z}\ \forall \gamma \in \Lambda_1\}.$$

Similarly,

$$\pi^{-1}(T_2) = \bigcup_{z \in \mathbb{Z}^k} S_z,$$

where

$$S_z = \{x \in \mathbb{R}^n \mid \gamma x = x + \gamma\tbinom{z}{0}\ \forall \gamma \in \Lambda_2\},$$

and

$$\pi^{-1}(T_2^N) = \bigcup_{z \in \mathbb{Z}^k} S_z^N,$$

where

$$S_z^N = \lambda_N S_z = \{x \in \mathbb{R}^n \mid \gamma x = x + \gamma \lambda_N \tbinom{z}{0}\ \forall \gamma \in \lambda_N \Lambda_2 \lambda_N^{-1}\}.$$

Likewise,

$$\pi^{-1}(\bar{T}_1) = \bigcup_{z \in \mathbb{Z}^\ell} \bar{R}_z,$$

where

$$\bar{R}_z \subset \{x \in \mathbb{R}^n \mid \tilde{\rho}(\gamma) x = x + \gamma\tbinom{0}{z}\ \forall \gamma \in \Lambda_1\}.$$

Set $\bar{T}_2^N = \rho(\lambda_N)\bar{T}_2$, which is an ℓ-torus of fixed points for $\rho(\lambda_N \Lambda_2 \lambda_N^{-1})$. Then

$$\pi^{-1}(\bar{T}_2^N) = \bigcup_{z \in \mathbb{Z}^k} \bar{S}_z^N,$$

where

$$\bar{S}_z^N = \tilde{\rho}(\lambda_N)\bar{S}_z \subset \{x \in \mathbb{R}^n \mid \tilde{\rho}(\gamma) x = x + \gamma \lambda_N \tbinom{0}{z}\ \forall \gamma \in \lambda_N \Lambda_1 \lambda_N^{-1}\}.$$

It is geometrically obvious that for any $N \in \mathbb{N}^+$ and $z \in \mathbb{Z}^k$ the topological ℓ-plane $\bar{R}_z$ and the topological k-plane $\bar{S}_0^N$ must have non-empty intersection; here is a simple proof:

Let $\varphi\colon R_z \to \mathbb{R}^n$ denote the restriction $\varphi = \tilde{h} \mid R_z$; φ maps R_z homeomorphically onto $\bar{R}_z$. Define $\psi\colon S_0^N \to \mathbb{R}^n$, $\psi = \tilde{\rho}(\lambda_N)\tilde{h}\lambda_N^{-1}$; ψ maps S_0^N homeomorphically onto $\bar{S}_0^N$. Since $\varphi(x+\zeta) = \varphi(x)+\zeta$ for every $x \in R_z$, $\zeta \in R_0 \cap \mathbb{Z}^n$ and $\psi(x+\zeta) = \psi(x)+\zeta$ for every $x \in S_0^N$, $\zeta \in S_0^N \cap \mathbb{Z}^n$, there exists $\delta > 0$ such that

$$(*) \qquad d(\varphi(x),x) < \delta \quad \forall x \in R_z \quad \text{and} \quad d(\psi(x),x) < \delta \quad \forall x \in S_0^N.$$

Now suppose that $\bar{R}_z \cap \bar{S}_0^N = \emptyset$. Since R_z and S_0^N are far apart outside a compact neighborhood of $R_z \cap S_0^N = \{(0,\dots,\frac{mz_1}{Nm^2+1},z_1,\dots,z_\ell)^t\}$, the same is true for $\bar{R}_z$ and $\bar{S}_0^N$, so that for small perturbations of φ and ψ, their images remain disjoint. Thus we may assume without loss of generality that φ and ψ are smooth.

Let $p\colon \mathbb{S}^n \to \mathbb{R}^n$ denote the stereographic projection, and set $R = p^{-1}R_z$, $\bar{R} = p^{-1}\bar{R}_z$, $S = p^{-1}S_0^N$, and $\bar{S} = p^{-1}\bar{S}_0^N$. R and S intersect transversally in two points: the original intersection point in $\mathbb{R}^n$ and the point at infinity. It follows from $(*)$ that R is homotopic to $\bar{R}$, S is homotopic to $\bar{S}$, and $\bar{R}$ and $\bar{S}$ intersect transversally at infinity. But $\bar{R}_z \cap \bar{S}_0^N = \emptyset$ implies that this is their unique point of intersection, which is a contradiction, because the mod 2 intersection number is a homotopy invariant.

For each $j = 0,\dots,Nm^2$, set $z_j = (j,0,\dots,0)^t \in \mathbb{Z}^\ell$ and

$$\tilde{q}_j = (0,\dots,\frac{mj}{Nm^2+1},\dots,0)^t,$$

so that $R_{z_j} \cap S_0^N = \{\tilde{q}_j\}$, $\tilde{q}_0 = 0$; then $\pi(\{q_j \mid 0 \le j \le Nm^2\}) = T_1 \cap T_2^N$ and we fix notation so that $\pi(\tilde{q}_j) = q_j$, and set $X_j = \bar{R}_{z_j} \cap \bar{S}_0^N \neq \emptyset$. We will show that X_j consists of a single point $X_j = \{\tilde{p}_j\}$ and that $\tilde{h}_i(\tilde{q}_j) = \tilde{p}_j$ for each i. Then $h_i(q_j) = h_i(\pi(\tilde{q}_j)) = \pi(\tilde{h}_i(\tilde{q}_j)) = \pi(\tilde{p}_j)$ for each i and the proof of (2.11) will be complete.

So suppose $x \in X_j = \bar{R}_{z_j} \cap \bar{S}_0^N$. Then for each $\gamma \in \Lambda_1$ we have $\tilde{\rho}(\gamma)x = x + \gamma\binom{0}{z_j}$ and for each $\gamma \in \lambda_N\Lambda_2\lambda_N^{-1}$ we have $\tilde{\rho}(\gamma)x = x$. Since Λ_1 and $\lambda_N\Lambda_2\lambda_N^{-1}$ together generate Ξ_N, it follows that there is a cocycle $\alpha\colon \Xi_N \to \mathbb{Z}^n$ such that $\tilde{\rho}(\gamma)x = x + \alpha(\gamma)$ for every $\gamma \in \Xi_N$.

Fix one of the generators γ_i. Then since γ_i has infinite order and Ξ_N has finite index in Γ, it follows that $\gamma_i^K \in \Xi_N$ for some $K \geq 1$. Now we have already ensured that the diffeomorphism $\rho(\gamma_i)$ is hyperbolic. In particular, the mapping $\rho(\gamma_i)^K - \mathrm{Id}\colon \mathbb{T}^n \to \mathbb{T}^n$ is non-singular at each point of $\mathbb{T}^n$, and $\tilde{\rho}(\gamma_i)^K - \mathrm{Id}\colon \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism.

Since $\tilde{\rho}(\gamma_i)^K - \mathrm{Id}$ is invertible, there is a unique point

$$\tilde{p}_j = (\tilde{\rho}(\gamma_i)^K - \mathrm{Id})^{-1}\alpha(\gamma_i^K) \in \mathbb{R}^n$$

such that $\tilde{\rho}(\gamma_i)^K \tilde{p}_j = \tilde{p}_j + \alpha(\gamma_i^K)$. Thus $X_j = \{\tilde{p}_j\}$. Moreover, $\tilde{q}_j$ is the unique point of $\mathbb{R}^n$ such that $\gamma_i^K \tilde{q}_j = \tilde{q}_j + \alpha(\gamma_i^K)$, hence

$$\begin{aligned}\tilde{\rho}(\gamma_i)^K \tilde{h}_i(\tilde{q}_j) &= \tilde{h}_i(\gamma_i^K(\tilde{q}_j)) = \tilde{h}_i(\tilde{q}_j + \alpha(\gamma_i^K)) \\ &= \tilde{h}_i(\tilde{q}_j) + \alpha(\gamma_i^K).\end{aligned}$$

Thus $\tilde{h}_i(\tilde{q}_j) = \tilde{p}_j$ and the proof of (2.11) is complete. ∎

COROLLARY 2.12: *$h_1(x) = \cdots = h_r(x)$ for every $x \in C$.*

Proof: $\bigcup_N (T_1 \cap T_2^N)$ is dense in C. ∎

COROLLARY 2.13: *Fix $\gamma_0 \in \Gamma$. Then $h_1(x) = \cdots = h_r(x)$ for every $x \in \gamma_0 C$.*

Proof: The subgroups $\gamma_0 \Lambda_1 \gamma_0^{-1}$ and $\gamma_0 \lambda_N \Lambda_2 \lambda_N^{-1} \gamma_0^{-1}$ together generate $\gamma_0 \Xi_N \gamma_0^{-1}$, which is again of finite index in Γ. Then the same argument that established (2.11) shows that $h_i(\gamma_0 q) = \rho(\gamma_0) h_i(q)$ for every $q \in T_1 \cap T_2^N$. ∎

Since $\bigcup_{\gamma \in \Gamma} \gamma C$ is dense in $\mathbb{T}^n$, this completes the proof of the topological version of (2.1).

Finally, we consider the case in which the existence of a fixed point is replaced by the more general hypothesis, existence of a finite orbit. We obtain the following

COROLLARY 2.14: *Suppose $\Gamma \subset \mathbf{SL}(n, \mathbb{Z})$ of finite index, $M = \mathbb{T}^n$, and $\rho \in R(\Gamma, \mathrm{Diff}(M))$ satisfies condition* (ii) *of* (2.1)*, together with*

(i′) *there exists a finite orbit, i.e., $x_0 \in M$ such that the set $\rho(\Gamma)x_0$ is finite.*

Again let $\rho_*\colon \Gamma \to \mathbf{GL}(n, \mathbb{Z})$ denote the homomorphism corresponding to the action on H_1. Then ρ is topologically equivalent to a rational affine action with linear part given by ρ_*; i.e., there exists a homeomorphism h of M, homotopic to the identity, such that $h^{-1}\rho(\gamma)h = \rho_*(\gamma) + \alpha(\gamma)$, where $\alpha\colon \Gamma \to \mathbb{Q}^n/\mathbb{Z}^n$ is a *cocycle in the coefficient module $\mathbb{Q}^n/\mathbb{Z}^n$ (rational points on $\mathbb{T}^n$) with Γ acting via ρ_*.*

Proof: Set Γ' equal to the largest normal subgroup of Γ which is contained in the stabilizer of the periodic point x_0. Since x_0 has finite orbit and Γ is finitely-generated, Γ' has finite index in Γ. Thus Γ satisfies the hypotheses of (2.1), and there exists $h \in \mathrm{Homeo}_1(M)$ such that $h^{-1}\rho(\gamma)h = \rho_*(\gamma)$ for every $\gamma \in \Gamma'$.

Fix hyperbolic generators $\{\gamma_1, \ldots, \gamma_r\}$ for the full group Γ by (5.2) of [K-L]. For each i, there exists $n_i \in \mathbb{N}^+$ such that $\gamma_i^{n_i} \in \Gamma'$. Consider the homeomorphism $\varphi_i = h^{-1}\rho(\gamma_i)h - h^{-1}\rho(\gamma_i)h(0)$. Since 0 is a fixed point for the action of Γ' under $h^{-1}\rho h$ and Γ' is normal in Γ, $h^{-1}\rho(\gamma_i)h(0)$ is again a fixed point for $h^{-1}\rho(\Gamma')h = \rho_*(\Gamma')$. Consequently, $h^{-1}\rho(\gamma_i)h(0) \in \mathbb{Q}^n/\mathbb{Z}^n$ is a rational point on the torus, and φ_i commutes with the linear Anosov diffeomorphism $\rho_*(\gamma_i^{n_i}) = h^{-1}\rho(\gamma_i^{n_i})h$. Since φ_i obviously fixes 0, it follows by (2.4) that $\varphi_i = \rho_*(\gamma_i)$. In other words, we have shown that $h^{-1}\rho(\gamma_i)h = \rho_*(\gamma_i) + \alpha(\gamma_i)$, where $\alpha(\gamma_i) = h^{-1}\rho(\gamma_i)h(0) \in \mathbb{Q}^n/\mathbb{Z}^n$. Since the γ_i generate Γ, this completes the proof. ∎

Note that the affine action $\rho_* + \alpha$ obtained in the preceding corollary is conjugate to the linear action ρ_* (by a translation) if and only if the cohomology class represented by the cocycle α is trivial, and the actions corresponding to different cocycles are equivalent (again by a translation) if and only if the cocycles are cohomologous. On the other hand, any non-trivial cohomology class in $H^1(\Gamma, \mathbb{Q}^n/\mathbb{Z}^n)$ (with Γ action given by ρ_*) gives rise to a non-linear affine action on $\mathbb{T}^n$. See [H3] for examples of such affine actions.

LEMMA 2.15: *Any cocycle $\alpha\colon \Gamma \to \mathbb{Q}^n/\mathbb{Z}^n$ (Γ acting via ρ_*) must vanish on a subgroup of finite index.*

Proof: Since Γ is finitely-generated, say by $\gamma_1, \ldots, \gamma_r$, there exists a common denominator m for the rational points $\alpha(\gamma_i)$. Thus the image of α is contained in the finite submodule $m^{-1}\mathbb{Z}^n/\mathbb{Z}^n \subset \mathbb{Q}^n/\mathbb{Z}^n$ (m-division points). ∎

Thus the "exotic" affine action corresponding to any cohomology class in $H^1(\Gamma, \mathbb{Q}^r/\mathbb{Z}^n)$ must restrict to a linear action on a subgroup of finite index, or in other words, must have a finite orbit. It will follow immediately from (3.1), below, that for actions with an Anosov element, topological and smooth equivalence coincide. Thus we will have shown

THEOREM 2.16: *The set of smooth equivalence classes of smooth actions $\rho \in R(\Gamma, \mathrm{Diff}(M))$ satisfying the hypotheses of (2.14), above, are indexed by the set of equivalence classes of representations $\rho_* \in \tilde{R}(\Gamma, \mathbf{GL}(n, \mathbb{Z}))$ which do not factor through a finite quotient (this set is finite by (2.5)) together with the cohomology group $H^1(\Gamma, \mathbb{Q}^n/\mathbb{Z}^n)$, with Γ acting on the coefficient module $\mathbb{Q}^n/\mathbb{Z}^n$ via ρ_*.*

3. Smooth conjugacy

There are several results to date which support the following general philosophy: for "sufficiently large" smooth group actions, topological and smooth equivalence coincide. (E.g., our result for Anosov actions of $\mathbb{Z}^{n-1}$ on $\mathbb{T}^n$ ([K-L], Theorem 4.12) and the recent theorem of Cawley [C] for non-Abelian groups generated by Anosov diffeomorphisms of $\mathbb{T}^2$.) In this section, we establish a similar result for Γ, generalizing "differential rigidity for Cartan actions," established in [H2]:

THEOREM 3.1: *Let Γ be a subgroup of finite index in $\mathbf{SL}(n, \mathbb{Z})$, $n \geq 3$, $M = \mathbb{T}^n$, and $\rho \in R(\Gamma, \mathrm{Diff}(M))$ such that*

(i) *ρ is topologically equivalent to the linear action corresponding to $\rho_*: \Gamma \to \mathbf{GL}(n, \mathbb{Z})$, the homomorphism given by the action on $H_1(M) \simeq \mathbb{Z}^n$, and*
(ii) *there exists $\gamma_0 \in \Gamma$ such that the diffeomorphism $\rho(\gamma_0)$ is Anosov.*

Then the topological conjugacy h between ρ and ρ_* must in fact be smooth.

Proof: Consider any rational point $p \in \mathbb{T}^n$. The orbit of p under the linear action $\rho_*(\Gamma)$ is finite, hence $x = h(p)$ has finite orbit under $\rho(\Gamma)$. Let Γ_x denote the stabilizer of x in Γ under ρ (which is the same as the stabilizer of p under ρ_*); Γ_x is a subgroup of finite index in Γ and hence in $\mathbf{SL}(n, \mathbb{Z})$.

LEMMA 3.2: *Let*

$$\varphi_x: \Gamma_x \to \mathbf{GL}(T_xM);\ \gamma \mapsto D_x\rho(\gamma)$$

denote the linear isotropy representation at the fixed point x. Then up to choice of coordinates in T_xM (conjugation in $\mathbf{GL}(n, \mathbb{R})$) in odd dimensions n the representation φ_x agrees with ρ_*, and, in even dimensions, the representations agree up to a factor of $\pm I$. (I.e., there exists $\psi_x: \mathbb{R}^n \to T_xM$ and a homomorphism $\iota_x: \Gamma_x \to \{\pm I\}$ such that for each $\gamma \in \Gamma_x$, $\varphi_x(\gamma) = \iota_x(\gamma)\psi_x\rho_*(\gamma)\psi_x^{-1}$, and when n is odd, ι_x must be trivial.)

Proof: First observe that since $\rho(\gamma_0)$ is Anosov, $\varphi_x(\gamma_0)$ is a hyperbolic linear transformation. Then the same argument which establishes (2.5) shows that under the appropriate choice of coordinates, φ_x coincides with either the standard or the contragredient representation on a subgroup of index at most two in Γ_x.

It follows that for any hyperbolic element $\gamma \in \Gamma_x$, $\varphi_x(\gamma)$ is hyperbolic, i.e., x is a hyperbolic fixed point for $\rho(\gamma)$. By the stable manifold theorem (e.g., Theorem III.7 in [Sh]), there are uniquely-determined, smooth, local stable and unstable manifolds for $\rho(\gamma)$ at x. Obviously, these manifolds must coincide with the global stable and unstable (topological) manifolds which are determined by the conjugacy. (In fact, since $\rho(\gamma)$ is topologically conjugate to $\rho_*(\gamma)$, the global stable and unstable manifolds through x are smooth; we can extend the smooth neighborhood in the unstable (stable) manifold by iterating $\rho(\gamma)$ $(\rho(\gamma)^{-1})$.) In any event, the unstable (stable) subspace in T_xM is tangent to the global unstable (stable) manifold through x, and the dimension of the unstable (stable) manifold, which is determined by ρ_*, is equal to the number of eigenvalues of $\varphi_x(\gamma)$ which lie outside (inside) the unit circle. Now any subgroup of finite index in $\mathbf{SL}(n,\mathbb{Z})$, $n \geq 3$, contains hyperbolic matrices for which the dimensions of the stable and unstable manifolds are unequal, and these dimensions are exchanged under the contragredient representation. Thus φ_x and ρ_* agree on a subgroup of index at most two.

Finally, for any diffeomorphism of an orientable manifold, the action on the orientation of the tangent space at any fixed point must coïncide with the action on the global orientation, so the actions of φ_x and ρ_* on orientation (determinants) agree. ∎

As a first application of (3.2), we obtain the following

PROPOSITION 3.3: *Let λ denote the Lebesgue measure on $M = \mathbb{T}^n$ (which is invariant under the action of $\mathbf{GL}(n,\mathbb{Z})$). Then under the hypotheses of (3.1), the ρ-invariant measure $\mu = h_*\lambda$ is given by a smooth positive density.*

Proof: (3.2) implies that for each periodic point x of period N for the Anosov diffeomorphism $f = \rho(\gamma_0)$, the determinant $\det(D_xf^N) = 1$. By [L] and [Ll-M-M], this implies that f preserves a smooth measure μ, given by a smooth, positive density.

Moreover, if we order the eigenvalues $\lambda_1, \ldots, \lambda_n$ for γ_0 (with multiplicity) so that $|\lambda_i| > 1$ for $1 \leq i \leq k$ and $|\lambda_i| < 1$ for $k+1 \leq i \leq n$, and denote by E^u the (invariant) unstable distribution for f, then the Jacobian of the restriction $|\det(D_xf^N|_{E^u})| = |\lambda_1 \cdots \lambda_k|^N$. By [L], this implies that the function

$\mathcal{J}^u\colon x \mapsto \log|\det(D_x f|_{E^u})|$ is cohomologous to a constant. Hence by [Si], the unique smooth invariant measure μ for f (which is equal to the Gibbs measure for the function $\mathcal{J}^u$) is equal to the Gibbs measure for the constant function. Hence μ is equal to the Bowen–Margulis measure, which is characterized as the Gibbs measure for the constant. For the linear Anosov diffeomorphism $\rho_*(\gamma_0)$, the Bowen–Margulis measure coincides with the Lebesgue measure λ. Since the Bowen–Margulis measure is a topological invariant, we conclude that $h_*\lambda = \mu$. ∎

We have already observed that the centralizer in $\mathbf{GL}(n,\mathbb{R})$ of any subgroup of finite index in $\mathbf{SL}(n,\mathbb{Z})$ is just the set of scalar matrices. Thus the isomorphisms ψ_x which appear in the statement of (3.2) are uniquely determined up to a scalar multiple. Under the identification of each tangent space T_xM with $\mathbb{R}^n$ via the standard (smooth) trivialization of $TM \simeq \mathbb{T}^n \times \mathbb{R}^n$, each ψ_x is identified with an element of $\mathbf{GL}(n,\mathbb{R})$. By the preceding discussion, this element is unique up to a scalar. Thus, by composing with the natural map $\mathbf{GL}(n,\mathbb{R}) \to \mathbf{PGL}(n,\mathbb{R})$; $g \mapsto \bar{g}$, we obtain a uniquely determined map

$$\eta\colon \mathcal{S} \to \mathbf{PGL}(n,\mathbb{R});\ x \mapsto \overline{\psi_x},$$

where $\mathcal{S} \subset M$ denotes the set of periodic points for $\rho(\Gamma)$, such that for every $x \in \mathcal{S}$ and every $\gamma \in \Gamma_x$,

$$\eta(x)\rho_*(\gamma) = \overline{\psi_x\rho_*(\gamma)} = \overline{D_x\rho(\gamma)\psi_x}$$

(where $D_x\rho(\gamma)\psi_x$ is identified with an element of $\mathbf{GL}(n,\mathbb{R})$ via the trivialization). The crux of the proof of (3.1) is to show that the presence of the Anosov diffeomorphism $\rho(\gamma_0)$ implies that the map η extends continuously to all of M.

Let $\mathbf{G}_k(n,\mathbb{R}) = \mathbf{G}_k(n,\mathbb{C})_{\mathbb{R}}$ denote the Grassman variety of k-planes in $\mathbb{R}^n$, $1 \le k \le n-1$. For each point $p_0 \in \mathbf{G}_k(n,\mathbb{R})$, the map

$$\mathbf{PGL}(n,\mathbb{C}) \to \mathbf{G}_k(n,\mathbb{C});\ \bar{g} \mapsto \bar{g}p_0$$

is $\mathbb{R}$-rational; given $\bar{g}_0 \in \mathbf{PGL}(n,\mathbb{R})$ and $p_1,\dots,p_\ell \in \mathbf{G}_k(n,\mathbb{R})$, the map

$$\mathbf{PGL}(n,\mathbb{R}) \to \mathbf{G}_k(n,\mathbb{R})^\ell = \underbrace{\mathbf{G}_k(n,\mathbb{R}) \times \cdots \times \mathbf{G}_k(n,\mathbb{R})}_{\ell \text{ times}};\quad \bar{g} \mapsto (\bar{g}p_1,\dots,\bar{g}p_\ell)$$

is a local imbedding on an open neighborhood of $\bar{g}_0$ (via the inverse function theorem) if and only if the intersection of the infinitesimal stabilizers

$$\bigcap_{i=1,\ldots,\ell} \mathfrak{pgl}(n,\mathbb{R})_{\bar{g}_0^{-1}p_i}$$

is zero. In particular, this condition is satisfied provided that the intersection of the stabilizers $\bigcap_{i=1,\ldots,\ell} \mathbf{PGL}(n,\mathbb{C})_{\bar{g}_0^{-1}p_i}$ is trivial.

Now suppose that the stable subspace for the hyperbolic linear automorphism $\rho_*(\gamma_0)$ is k-dimensional, and set $p_1 \in \mathbf{G}_k(n,\mathbb{R})$ equal to the corresponding point in the Grassman variety. By the Borel density theorem, the intersection

$$\bigcap_{\gamma\in\Gamma} \mathbf{PGL}(n,\mathbb{C})_{\gamma p_1} = \bigcap_{\gamma\in\Gamma} \gamma(\mathbf{PGL}(n,\mathbb{C})_{p_1})\gamma^{-1}$$

is a (proper) normal subgroup of $\mathbf{PGL}(n,\mathbb{C})$, hence is trivial. Thus we can choose matrices $\sigma_1 = I, \sigma_2, \ldots, \sigma_\ell \in \Gamma$ such that with $p_i = \sigma_i p_1$, which is the stable subspace for $\rho_*(\gamma_i) = \rho_*(\sigma_i\gamma_0\sigma_i^{-1})$, the stabilizer of the point $p = (p_1, \ldots, p_\ell) \in \mathbf{G}_k(n,\mathbb{R})^\ell$ in $\mathbf{PGL}(n,\mathbb{C})$ is trivial. It follows from a general theorem of Borel–Serre [B-S] (cf. 3.1.3 of [Z2]) that the $\mathbf{PGL}(n,\mathbb{R})$ orbit of p in the product variety $\mathbf{G}_k(n,\mathbb{R})^\ell$ is locally closed in the Hausdorff topology. Set $C \subset \mathbf{G}_k(n,\mathbb{R})^\ell$ equal to the closure of this orbit, and let $U \subset C$ denote the orbit itself, which is open in C. Note that by the preceding paragraph, for every $\bar{g}_0 \in \mathbf{PGL}(n,\mathbb{R})$, the map $\bar{g} \mapsto \bar{g}q$ is a local imbedding on a neighborhood of $\bar{g}_0$ in $\mathbf{GL}(n,\mathbb{R})$, where $q = (q_1, \ldots, q_\ell) = (\bar{g}_0 p_1, \ldots, \bar{g}_0 p_\ell) = \bar{g}_0 p \in U$.

For $1 \le i \le \ell$, let $q_i\colon M \to \mathbf{G}_k(n,\mathbb{R})$ denote the map corresponding to the stable distribution for the Anosov diffeomorphism $\rho(\gamma_i) = \rho(\sigma_i)\rho(\gamma_0)\rho(\sigma_i)^{-1}$ under the identification of TM with $M \times \mathbb{R}^n$ via the standard trivialization. Now suppose $x \in \mathcal{S}$. It follows from the definitions that $q_i(x) = \eta(x)p_i$. In particular, $q(x) = (q_1(x), \ldots, q_\ell(x)) \in U$.

Since each of the maps q_i is continuous (in the Hausdorff topology on $\mathbf{G}_k(n,\mathbb{R})$), it will follow that η extends to a continuous map $M \to \mathbf{PGL}(n,\mathbb{R})$ simply by mapping $x \in M$ to the unique point $\bar{g} \in \mathbf{PGL}(n,\mathbb{R})$ such that $\bar{g}p = q(x)$, provided that this makes sense, i.e., provided that $q(x) \in U$ for every $x \in M$. We have already seen that this is true at every $x \in \mathcal{S}$. Since $\mathcal{S}$ is dense in M and C is closed, it follows by continuity that $q(x) \in C$ for every $x \in M$. Then since U is open in C, the set $\mathcal{R} = \{x \in M \mid q(x) = (q_1(x), \ldots, q_\ell(x)) \in U\}$ is open in

M. Moreover, for any element $\gamma \in \Gamma$ and any $x \in \mathcal{S}$,

$$D_x\rho(\gamma)q(x) = D_x\rho(\gamma)\eta(x)p = \eta(\rho(\gamma)x)(\rho_*(\gamma)p),$$

where, as usual, we identify $D_x\rho(\gamma)$ with an element of $\mathbf{GL}(n,\mathbb{R})$ via the standard trivialization. (In fact, we have only proved this for $\gamma \in \Gamma_x$. But if $x, \rho(\gamma)x \in \mathcal{S}$, the derivative $D_x\rho(\gamma)$ must intertwine the isotropy representation of Γ_x at x with that of $\Gamma_{\rho(\gamma)x} = \gamma\Gamma_x\gamma^{-1}$ at $\rho(\gamma)x$, and we may again make use of the fact that the centralizer of Γ_x in $\mathbf{GL}(n,\mathbb{R})$ consists only of scalars.) Thus if $x \in \mathcal{R}$, so that $q(x) = \bar{g}p$ for some $\bar{g} \in \mathbf{PGL}(n,\mathbb{R})$, and $\gamma \in \Gamma$, then $q(\rho(\gamma)x) = \bar{g}'p \in U$, where $g' = D_x\rho(\gamma)g\rho_*(\gamma)^{-1}$, by continuity. In particular, $\rho(\gamma)x \in \mathcal{R}$, and the set $\mathcal{R}$ is invariant under the action of Γ. In other words, the complement $M - \mathcal{R}$ is a closed, Γ-invariant set. Now apply the following obvious fact; for the sake of completeness, we sketch a simple proof.

LEMMA 3.4: *The only closed, Γ-invariant subsets of M are finite unions of finite orbits.*

Proof: Obviously this property is invariant under topological conjugacy, hence it will suffice to establish it for the linear action ρ_*. Suppose such a set contains a point x with infinite orbit, i.e., a point with at least one irrational coordinate, say x_i. Considering the action of an appropriate unipotent matrix in Γ, we see that the closure of the orbit contains the circle $\mathcal{C} = \{(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n) \mid t \in \mathbb{R}\}$. Since Γ acts minimally on $\mathbb{P}^{n-1}$, the orbit of $\mathcal{C}$ is dense in $\mathbb{T}^n$. ∎

Thus $\mathcal{R} = M$, η extends to a continuous map η: $M \to \mathbf{PGL}(n,\mathbb{R})$, and we obtain the following

LEMMA 3.5: *For every hyperbolic matrix $\gamma \in \Gamma$, the diffeomorphism $\rho(\gamma)$ is Anosov.*

Proof: Let ω denote the smooth Γ-invariant volume form on M which exists by (3.3). Under the identification of TM with $M \times \mathbb{R}^n$, each ω_x, $x \in M$, is identified with a map ω_x: $\mathbf{GL}(n,\mathbb{R}) \to \mathbb{R}$. When n is odd, the equation $\omega_x(\tilde{\eta}(x)) = 1$ uniquely determines a continuous lift of η to $\tilde{\eta}$: $M \to \mathbf{GL}(n,\mathbb{R})$; when n is even, $\tilde{\eta}$: $M \to \mathbf{GL}(n,\mathbb{R})/\{\pm I\}$ is only defined up to a choice of sign. In any event, modulo $\{\pm I\}$, $\tilde{\eta}$ gives a continuous equivalence between the derivative cocycle for the action of Γ and the constant cocycle corresponding to the linear representation ρ_*. Thus the images of the stable and unstable subspaces for the linear Anosov map $\rho_*(\gamma)$ under $\tilde{\eta}$ exhibit continuous stable and unstable distributions for $\rho(\gamma)$ explicitly. ∎

We are now in a position to apply the argument of Section 4 in [K-L]. Fix an Abelian subgroup $\mathcal{A} \subset \Gamma$ of rank $n-1$; $\rho_*(\mathcal{A})$ consists entirely of hyperbolic linear automorphisms, hence by (3.5) $\rho(\mathcal{A})$ consists entirely of Anosov diffeomorphisms. According to Theorem 4.12 of [K-L], the restriction of this action to a subgroup of finite index is smoothly conjugate to ρ_*. (In fact, since we have already observed that the conjugate action ρ preserves the finite-dimensional linear data at each periodic point, and in particular the periodic exponents, we need not appeal to [K-L], but may instead refer to [H2].) Since the conjugacy is unique up to a rational translation by (2.4), it follows that the homeomorphism h conjugating ρ to ρ_* is smooth. ∎

It is probably worth remarking that rather than invoking either [K-L] or [H2], we can avoid reference to the subgroup $\mathcal{A}$ altogether and conclude directly that the continuous equivalence $\tilde{\eta}$ constructed in the proof of (3.5) is in fact smooth. This follows from the fact that among the Anosov diffeomorphisms $\rho(\gamma)$, $\gamma \in \Gamma$, there are sufficiently many with C^1 stable and unstable foliations (e.g., those of codimension-1). This implies that the images of constant vector fields under the equivalence $\tilde{\eta}$ are in fact C^1, hence uniquely integrable. It is easy to see that those vector fields commute and hence define a Γ-invariant, C^1, flat, affine structure on M. Using regularity results for Anosov maps, it follows that this structure is C^∞. Now it is easy to see that the structure must be conjugate to the standard one, and hence ρ is smoothly equivalent to an affine action. Since ρ has a fixed point, the equivalent affine action is linear.

4. A new construction of lattice actions and some open problems

Our principal goal in this section is to describe some "new" smooth (in fact, real-analytic), volume-preserving, ergodic actions of higher-rank lattices on compact manifolds. Although the constructions are all quite elementary and the examples appear (at least in retrospect) to be quite natural, they are not among the algebraic examples listed by Zimmer [Z1], and, to the best of our knowledge, they have not been discussed previously in this context.

The basic construction is a simple variant of the well-known blowing-up procedure from algebraic geometry. In particular, we recall, very briefly, how to blow up the origin in $\mathbb{A}^n = \mathbb{A}^n(\mathbb{F})$, $n \geq 2$. We define a subvariety $B \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ as follows. Let $x = (x_1, \dots, x_n)$ denote the usual affine coordinate on $\mathbb{A}^n$ and $[y] = [y_1, \dots, y_n]$ denote the homogeneous coordinate on $\mathbb{P}^{n-1}$, so that $[\lambda y] = [y]$ for all $\lambda \in \mathbb{F}$, $y \in \mathbb{F}^n - \{0\}$. Set

$$B = \{(x, [y]) \in \mathbb{A}^n \times \mathbb{P}^{n-1} \mid x_i y_j = x_j y_i \quad \forall 1 \leq i, j \leq n\}.$$

Let $\pi\colon B \to \mathbb{A}^n$ denote the map obtained by restriction from the projection onto the x coordinate.

The following properties of B and π are easily verified. B is a smooth subvariety of $\mathbb{A}^n \times \mathbb{P}^{n-1}$, hence π is obviously regular. For each $x \in \mathbb{A}^n - \{0\}$, the preimage $\pi^{-1}(x)$ is a single point, and $\pi^{-1}(0) \simeq \mathbb{P}^{n-1}$. In fact, the restriction of π to $B - \pi^{-1}(0)$ is an isomophism onto $\mathbb{A}^n - \{0\}$, so that B is obtained by "blowing up" the origin in $\mathbb{A}^n$ to a copy of $\mathbb{P}^{n-1}$. The standard linear action of $\mathbf{GL}(n, \mathbb{F})$ on $\mathbb{A}^n \times \mathbb{P}^{n-1}$ leaves B invariant, hence we obtain an action of $\mathbf{GL}(n, \mathbb{F})$ on B by isomorphisms.

Now specialize to the case $\mathbb{F} = \mathbb{R}$, and restrict the action of $\mathbf{GL}(n, \mathbb{R})$ to $\mathbf{GL}(n, \mathbb{Z})$. Replace the variety $\mathbb{A}^n = \mathbb{R}^n$ in the preceding construction by $M = \mathbb{T}^n$. Then we obtain a smooth real subvariety $\tilde{M}$ of the compact real variety $\mathbb{T}^n \times \mathbb{P}^{n-1}$ on which $\mathbf{GL}(n, \mathbb{Z})$ acts by isomorphisms, together with a $\mathbf{GL}(n, \mathbb{Z})$-equivariant regular morphism $\pi\colon \tilde{M} \to M$ which restricts to an isomorphism of the (Zariski-) open set $U = \tilde{M} - \pi^{-1}(0)$ onto $\mathbb{T}^n - \{0\}$. Obviously the action of $\mathbf{SL}(n, \mathbb{Z})$ preserves a smooth n-form on $\tilde{M}$, namely, the pullback of the standard volume form on $\mathbb{T}^n$ under π. However, an easy calculation shows that the Jacobian of π is zero along the exceptional divisor $D = \pi^{-1}(0)$.

To remedy this situation, and produce an example of an action which preserves a smooth (non-vanishing) volume form, we simply adjust the smooth structure in a neighborhood of D. The new structure which we introduce will in fact be real analytic, and with respect to this structure the action of $\mathbf{GL}(n, \mathbb{Z})$ will again be analytic. Also, the new structure coincides with the usual one on $U = \tilde{M} - D$.

The construction is purely local, and we shall in fact describe a real analytic structure on $B \subset \mathbb{R}^n \times \mathbb{P}^{n-1}(\mathbb{R})$ with respect to which the action of $\mathbf{SL}(n, \mathbb{R})$ is analytic and preserves a smooth volume. In the context of the preceding discussion, we can give the following brief description. Begin by defining an exotic analytic structure on $\mathbb{R}^n$ by replacing the standard coordinate x with a new coordinate $x' = ||x||^{n-1}x$. Note that this structure is obviously equivalent to the old one on $\mathbb{R}^n - \{0\}$.

With respect to this structure on $\mathbb{R}^n$, the action of $\mathbf{GL}(n,\mathbb{R})$ is no longer smooth at the origin, and the invariant density blows up there. However, it's easy to check that with respect to the corresponding coordinates on B, the action of each linear transformation is analytic, and the invariant density is smooth and non-vanishing.*

It will prove convenient for what follows to give an alternative construction of $\tilde{M}$ with its "exotic" analytic structure. Begin by observing that the map

$$\varphi\colon \mathbb{R}^n - \{0\} \to X = \{x \in \mathbb{R}^n \mid ||x|| > 1\};\ x \mapsto \frac{(||x||^n + 1)^{1/n}}{||x||}x$$

has Jacobian 1 with respect to the standard coordinates on $\mathbb{R}^n$. If $A \in \mathbf{GL}(n,\mathbb{R})$ is any linear transformation on $\mathbb{R}^n$, then the diffeomorphism $\varphi A \varphi^{-1}$ of X extends analytically to a neighborhood of the boundary (although the neighborhood will in general depend on A); this is easily seen directly from the formula

$$\varphi A \varphi^{-1}(x) = \left(\frac{||x||^n - 1}{||x||^n} + \frac{1}{||Ax||^n}\right)^{1/n} Ax.$$

Let (r,θ) denote the standard polar coordinates on X, $X = \{(r,\theta) \mid r > 1, \theta \in \mathbb{S}^{n-1}\}$, and define new coordinates (s,θ) by setting $s = r^n - 1$. Note that the (s,θ) coordinates extend analytically across the boundary and have the property that the standard volume form is proportional to $ds \wedge d\theta$. Now define B as the quotient of $\bar{X}$ under the identification of antipodal points on the boundary. Let $\pi\colon X \to B$ denote the quotient map, and cover B with n neighborhoods $U_i = \{\pi(x) \mid x \in \bar{X}, x_i \neq 0\}$. Define analytic coördinates (t,θ) on U_i by setting $t = s$ on $U_i^+ = \{\pi(x) \in U_i \mid x_i > 0\}$ and setting $t = -s$ on $U_i^- = \{\pi(x) \in U_i \mid x_i < 0\}$.

At this point it is clear that the action of each $A \in \mathbf{SL}(n,\mathbb{R})$ is analytic on the dense open set $B - D$, where $D = \pi(\partial\bar{X})$, and preserves the volume form $ds \wedge d\theta$. To see that the action is analytic near D, note that the map corresponding to A has the form

$$(s,\theta) \mapsto (f(s,\theta), g(\theta)),$$

where g corresponds to the standard action of A on $\mathbb{P}^{n-1}$, which is certainly analytic. Preservation of the volume form $ds \wedge d\theta$ forces

$$\frac{\partial f}{\partial s} = \left|\frac{\partial g}{\partial \theta}\right|^{-1} \quad \text{(the inverse Jacobian)},$$

* To be precise: The standard structure on B is obtained by covering B with n neighborhoods $U_i = \{(x,[y]) \in B \mid y_i \neq 0\}$ and taking as coordinates on U_i the n variables $z_1, \ldots, x_i, \ldots, z_n$, where $z_j = y_j/y_i$. The new structure will be obtained in the same way, but with x_i' in place of x_i.

hence

$$f(s,\theta) = \left|\frac{\partial g}{\partial \theta}\right|^{-1} s,$$

so that the action of A glues analytically across the boundary.

As we have already observed, the above construction is purely local. Thus it can be applied at any fixed point, p, of a volume-preserving action which is smoothly equivalent to a linear action in a neighborhood of p. In particular, this is true for any action from the list of algebraic examples described in the introduction which has a fixed point. It is also worth remarking that the construction is fairly general, in as much as by arithmeticity, any irreducible higher-rank lattice Γ has a finite extension which has a subgroup of finite index Γ' which can be faithfully represented in $\mathbf{SL}(n,\mathbb{Z})$ (for some large n), and the construction can be applied to the action of Γ' on the torus.

Now suppose that Γ is a subgroup of finite index in $\mathbf{SL}(n,\mathbb{Z})$ with multiple fixed points on the torus. The preceding construction of $\tilde{M}$ makes it clear that instead of blowing up a single point to a projective space, we can just as well blow up multiple fixed points (or appropriately chosen points along finite orbits) to spheres and glue the boundary spheres together in pairs (either with or without the antipodal map) to obtain additional examples of real-analytic, ergodic, volume-preserving actions on manifolds with complicated topology.

Observe that the actions which we have constructed are not locally C^1-rigid. For the sake of concreteness, take the simplest example, that is let $\Gamma = \mathbf{SL}(n,\mathbb{Z})$ and $\sigma\colon \Gamma \to \operatorname{Diff} \tilde{M}$ denote the volume-preserving action obtained by blowing up the origin on the torus as described above. We construct a smooth one-parameter family σ_α of smooth (C^∞ but not real analytic) actions such that $\sigma = \sigma_1$, all actions σ_α are topologically conjugate, but no two of them are C^1 conjugate.

Fix a neighborhood V of the exceptional divisor D in $\tilde{M}$. For each positive real number α, define $h_\alpha\colon \tilde{M} \to \tilde{M}$ so that, with respect to the (s,θ) coordinates defined above,

$$h_\alpha(s,\theta) = (s^\alpha, \theta) \quad \forall (s,\theta) \in V,$$

and so that the restrictions $h_\alpha \mid U$ define a smooth path of C^∞ diffeomorphisms on the open set $U = \tilde{M} - D$ (with respect to the usual Fréchet manifold structure on Diff U). Now set $\sigma_\alpha = h_\alpha^{-1}\sigma h_\alpha$. Inside V, each linear map A takes the form

$$A(s,\theta) = \left(\left|\frac{\partial g}{\partial \theta}\right|^{-\alpha} s, g(\theta)\right)$$

with respect to the h_α coordinates. Hence the action of each $\gamma \in \Gamma$ under σ_α is smooth on V and varies smoothly with α. The same is true by construction on U. Obviously, all the actions σ_α are topologically equivalent. However, the unique absolutely continuous invariant measure for σ_α is given by a smooth positive density on U, and on V has the form $s^{\frac{1}{\alpha}-1} ds \wedge d\theta$. For $\alpha < 1$ the density vanishes at D, while $\alpha > 1$ it blows up at D, and the order of the density at D is different for each α, so that no two of the actions σ_α are C^1-conjugate. (It is perhaps worth noticing that the algebraic action corresponding to the standard blowing-up construction is C^∞ conjugate but not analytically conjugate to the action $\sigma_{1/n}$.)

Two questions arise immediately.

QUESTION 1: *Is the action σ C^1 (or C^∞) structurally stable (i.e., are nearby actions topologically conjugate)?*

QUESTION 2: *Is σ locally rigid in the category of volume-preserving actions (i.e., are nearby volume-preserving actions smoothly conjugate)?*

Since the restriction of σ to the projective space $D \subset \tilde{M}$ is simply the standard projective action τ on $\mathbb{P}^{n-1}$ (which is algebraic but not volume-preserving) it would appear that questions about the rigidity of σ may be closely related to questions about the rigidity of τ. More generally this suggests that it may be necessary to consider non-volume-preserving actions in order to understand the volume-preserving ones.

Another interesting open question is whether the pathology exhibited by our construction can be eliminated by requiring that the action preserve a stronger geometric structure than volume. For example, take $\Gamma = \mathbf{SP}(2n, \mathbb{Z})$. The standard symplectic form on $\mathbb{T}^{2n}$ is the unique symplectic form preserved by the standard linear action of Γ. It is easy to see that for $n \geq 2$, this form blows up at D under our construction.

QUESTION 3: *Does there exist a smooth symplectic action of $\mathbf{SP}(2n, \mathbb{Z})$ on a compact $2n$-dimensional symplectic manifold which is not smoothly conjugate to an affine action?*

Probably the most obvious question is whether anything like the effect of our construction can be achieved without changing the topology.

QUESTION 4: *Is every smooth, volume-preserving, ergodic action of* $\mathbf{SL}(n,\mathbb{Z})$ *on* $\mathbb{T}^n$, $n \geq 3$, *which induces the standard action on homology smoothly conjugate to an affine action?*

Finally, the "exotic" volume-preserving actions of subgroups of finite-index in $\mathbf{SL}(n,\mathbb{Z})$ on $\tilde{M}$ which we describe above share with the standard actions on $\mathbb{T}^n$ the property that the only ergodic invariant measures are either atomic or absolutely continuous.

QUESTION 5: *Let* $\Gamma \subset \mathbf{SL}(n,\mathbb{Z})$, $n \geq 3$, *be a subgroup of finite index, and let* ρ *be a smooth action of* Γ *on an* n*-dimensional manifold. Is every ergodic invariant measure either atomic or absolutely continuous?*

These questions represent only a small sample of the interesting open problems related to the classification of smooth actions of higher-rank lattices on compact manifolds.

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